Random walks and Lévy flights observed in fluid flows

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Abstract

We use a rapidly rotating tank filled with water to study the random walks of tracer particles in two different flows: one with coherent structures (vortices and jets), and one without (weakly turbulent). Most random walks, such as those taken by dye molecules diffusing by Brownian motion, obey the Central Limit Theorem; their motion can be characterized by a diffusion constant. The random walks in our experiment with coherent structures are Lévy flights: random walks with infinite mean square step size. Lévy flights are particularly interesting as the Central Limit Theorem does not apply to them; the effective diffusion constant becomes infinite. Tracer particles in the weakly turbulent flow diffuse normally. We discuss our observations and mathematical descriptions of transport in situations with Lévy flights.

I. INTRODUCTION

Mathematicians have been interested in random walks for centuries [1]; since Einstein's 1905 paper on Brownian motion [2], random walks have been used as a tool for science as well. In his paper, Einstein showed that the diffusion of dye in water due to Brownian motion arose from the random thermal motion of the dye and water molecules. The path of a dye molecule can be modeled as a random walk with straight line trajectories between collisions with other molecules (of water or dye); following the collision the dye molecule moves in a new direction until the next collision. The microscopic random walks of dye molecules results in the diffusive spreading of dye, a macroscopic result.

These simple ideas can be applied to many physical systems. In introductory physics courses, students learn that current carried by wires is related to the random motion of electrons in the wire. When a voltage is applied to a wire, the random collisions of the electrons are biased by an electric field, and their biased random walks produce the overall current carried by the wire. In stirred fluid systems, the stirring action is typically far more important than the Brownian motion of the individual molecules in mixing dye to all parts of the system. When this stirring is random such as in turbulent flows, the mixing can be understood as arising from random walks on a macroscopic (rather than microscopic) scale [3–6]. For example, in oceans, typical length scales are $L \approx 10$ km. For tracers diffusing with a typical molecular diffusion constant of $D \approx 10^{-5}$ cm^2/s, the molecular diffusive time scale for Brownian motion would be $L^2/D \approx 10^9$ yr. However, mixing in the ocean occurs on time scales of 1–10 yr, due to the large scale flows.

The connection between the mathematical ideas of random walks and the physical concepts of diffusion is made through the Central Limit Theorem [8]. Mathematically, the effects of diffusion are measured by the growth in time of the variance σ^2 of the collection of randomly walking particles,

$$\sigma^2(t) = \langle x^2(t) \rangle - \langle x(t) \rangle^2 = 2Dt, \tag{1}$$

where D is the diffusion constant, and the angle brackets $\langle . \rangle$ represent averages over the positions x of the random walkers. When the random walk is symmetric, $\langle x(t) \rangle = 0$ and the variance is given by $\sigma^2(t) = \langle x^2(t) \rangle$, the mean square displacement. The square root of the variance σ is a characteristic size of the collection of random walkers, and D determines the rate of growth of this characteristic size. The Central Limit Theorem defines D in terms of the motions of the individual random walkers [7]. Consider the individual steps taken by a random walker, that is, the distance moved between direction changes. These steps have a random length, and there exists a *probability distribution function* (PDF) for these lengths, $P(l)$. The probability of taking a step with length between l and $l + dl$ is given by $P(l)dl$. The Central Limit Theorem then shows that the diffusion constant is given by

$$D = \frac{\langle l^2 \rangle - \langle l \rangle^2}{2T}, \tag{2}$$

where T is a characteristic time between steps, and the moments of l are taken from $P(l)$:

$$\langle l^n \rangle = \int_{-\infty}^{\infty} dl \; l^n P(l). \tag{3}$$

While $P(l)$ may be a complicated function, the only connection this PDF has with the diffusion constant is through its first two moments, $\langle l \rangle$ and $\langle l^2 \rangle$. Note that this result applies to higher dimensions with the variance being defined as $\sigma^2 = \langle \vec{x} \cdot \vec{x} \rangle - \langle \vec{x} \rangle \cdot \langle \vec{x} \rangle$, and l is the length of a single step.

In the past two decades, physicists have become interested in random walks where the step length PDF has an infinite second moment, that is, $\langle l^2 \rangle = \infty$ [8,9]. In this case, the Central Limit Theorem no longer applies and the diffusion constant is undefined. Such random walks with infinite second moments were first studied by the mathematicians, in particular Paul Lévy in the 1930s [10]. These special random walks are now termed Lévy flights [9]. If the probability distribution function decays as a power law, $P(l) \sim l^{-\mu}$, then $\langle l^2 \rangle = \infty$ if $\mu < 3$ (the integral in Eq. (3) diverges). In these cases, the variance grows faster than linearly in time: $\sigma^2(t) \sim t^\gamma$, with $1 < \gamma < 2$; this is *superdiffusive* motion. If $0 < \gamma < 1$, the motion is *subdiffusive*. Normal diffusion is the case $\gamma = 1$, while anomalous diffusion is $\gamma \neq 1$.

An example of a normal random walk is shown in Fig. 1(a,b), and a superdiffusive random walk is shown in Fig. 1(c,d). In both cases, $P(l) \sim l^{-\mu}$ for large l; for the normal random walk $\mu = 3.8 > 3$ and for the superdiffusive random walk $\mu = 2.2 < 3$. For the normal random walk, after many steps, the overall motion is the average effect of all of the steps. For the anomalous case, the overall motion is dominated by a few rare steps. This is the signature of anomalous random walks: the motion is always determined by a few steps, no matter how many steps have been taken.

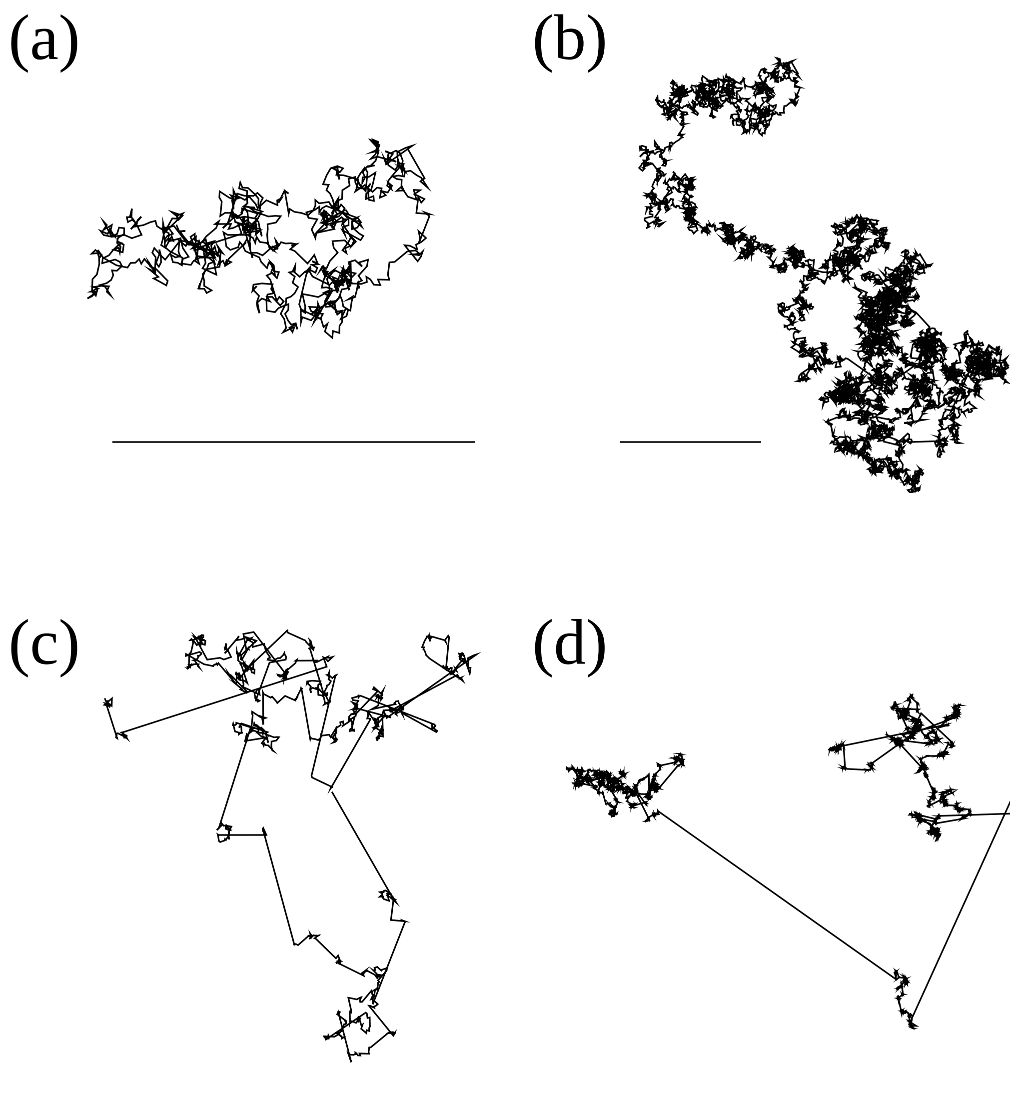

FIG. 1. Random walk leading to normal diffusion (a) after 1000 steps; (b) after 10000 steps; random walk (Lévy flight) leading to superdiffusion (c) after 1000 steps; (d) after 10000 steps. Note the self-similar appearance of (c) and (d); long steps can be seen at both scales. For both random walks, the mean step size $\langle l \rangle = 1$, the angle is chosen randomly for each step, and the step length is chosen from a probability distribution that decays as $P(l) \sim l^{-\mu}$ for large l. For (a) and (b), the step size distribution has a decay exponent of $\mu = 3.8$, and for (c) and (d) $\mu = 2.2$. The horizontal bars are of length 50. These random walks were generated by artificially constructing PDFs with the correct power law decay [11].

Lévy flights were first used by physicists to explain experimental observations of photoconductivity in amorphous materials [12,13]. Several experiments since then have found anomalous diffusion. Subdiffusion was found in experiments with arrays of vortices [14,15], while particles on the surface of a vertically shaken liquid were observed to move superdiffusively [16]. Anomalous diffusion has also been found in ocean tracers [17], the motion on the surface of human skin fibroblasts [18], and in mixing of polymer-like micelles [19]. Additionally, Lévy flights have proven extremely useful for understanding Hamiltonian systems [9,20–23]

In Sec. II, we describe our experimental observations of Lévy flights in two quasi-two-dimensional flows, one an ordered flow with vortices and jets, the other a weakly turbulent flow with no long-lived coherent structures [3–6]. In Sec. III, we discuss the theory linking the random walk flight PDFs with the anomalous diffusion exponent γ [6].

II. EXPERIMENTS

We study random walks in a rapidly rotating tank completely filled with distilled water. The tank has an annular shape, with an inner radius of 18.9 cm, an outer radius 43.2 cm, and a depth ranging from 17.1 cm at the inner cylinder to 20.3 cm at the outer cylinder. The gentle slope in the bottom of the annulus is used to model the curvature of planets for previous geophysical experiments [24], but is not important for the current work. The lid is flat and made from Plexiglas to allow viewing of the flow from above. The annulus rotates rigidly at 1.0 Hz (6.28 rad/s), and a video camera rotates above the annulus at the same speed. A consequence of this rapid rotation is that the flow becomes two-dimensional, with no vertical motion, as predicted by the Taylor-Proudman theorem [25]. Velocity probes at the top and bottom of the experiment (20 cm apart) are 95% to 99% correlated for our experiments [24]. (The velocity probes are 1 cm away from the tank surfaces, well outside of the Ekman boundary layers, which are only about 0.05 cm thick.)

We add several hundred small wax tracer particles to visualize the flow. Using the video camera and an automated tracking system, we follow the motion of individual particles for times up to 1000 s [26]. We examine the individual trajectories of particles, and consider the behavior of all of the particles as a collection of random walks. The tracer particles are illuminated in a horizontal slice through the middle of the tank, far from the boundary layers present at the top and bottom surfaces.

By pumping fluid through small holes in the sloped bottom, we stir the fluid inside the annulus. The three-dimensional nature of the pumping is confined to a thin boundary layer (the Ekman layer) at the bottom of the annulus; this layer is less than 1 mm thick, and the particles seen by the camera are in the bulk fluid which is moving two-dimensionally. (An additional Ekman layer is at the top of the annulus, between the moving fluid and the rigid lid.) The pumping causes an ordered flow with four large vortices which slowly rotate around the annulus [see Fig. 2(a)]. In the co-rotating reference frame of the four vortices, the outer fluid is moving in a counter-clockwise jet. (For exact details of the forcing resulting in this flow, see Ref. [6].)

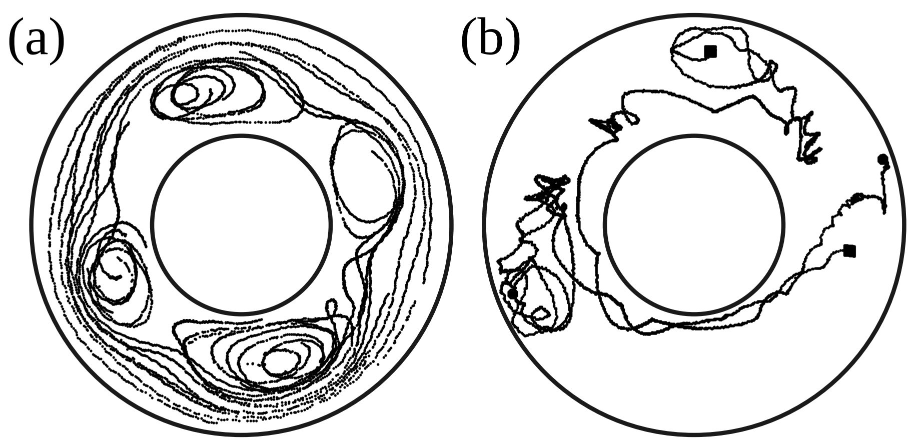

FIG. 2. (a) Several tracer particles reveal an ordered flow with four vortices and a counter-clockwise flowing jet encircling the vortices. This flow is shown in a reference frame co-rotating with the four vortices. (b) Two tracer particle trajectories in a weakly turbulent flow show the lack of long-lived coherent structures. The ends of one particle trajectory are marked by circles, the other by squares, and both particles start their trajectories at the right side of the annulus.

Figure 3 shows two particle trajectories in the ordered flow. It can be seen that the vortices move back and forth relative to each other. The tracer particles follow complicated trajectories, alternately spending time in one of the vortices and then switching into the outer jet. When they are in the jet, they travel long distances around the annulus before being captured again by a vortex. Figure 4(a,b) shows the angular position of these two particles as a function of time. The long diagonal lines are the flights in the jet.

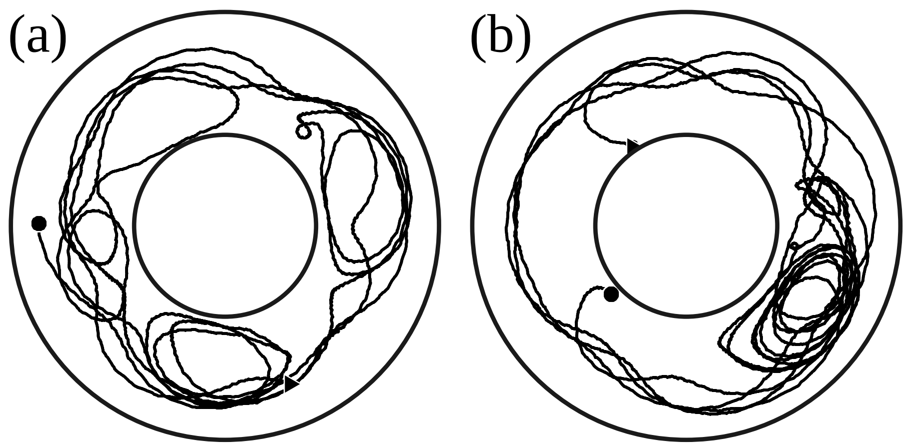

FIG. 3. (a) and (b): two trajectories from the ordered flow shown in Fig. 2(a). The beginning of each trajectory is marked with a circle, the end with a triangle.

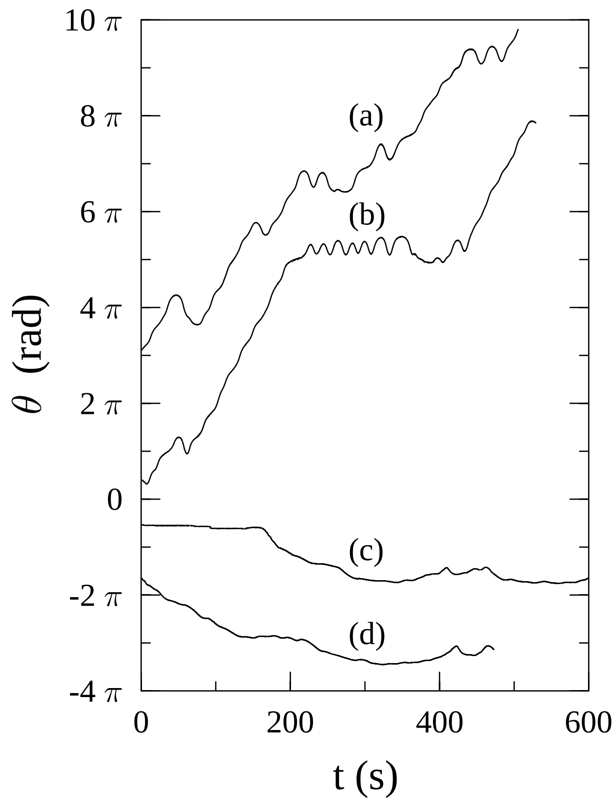

FIG. 4. The angular position of tracer particles. (a) and (b) correspond to the two trajectories shown in Fig. 3(a,b) respectively. The diagonal lines are flights in the jet, while the small wavy oscillations occur when the particles are caught in a vortex. (c) corresponds to the trajectory marked by circles in Fig. 2(b), and (d) to the trajectory marked by squares.

By examining several hundred trajectories, the length of flights can be measured and the PDF for flight lengths can be determined, as shown in Fig. 5(a). The PDF shows clear power law decay, with a decay exponent $\mu = 2.3 \pm 0.1$. Since $\mu < 3$, the second moment of this distribution function is infinite, and the particle trajectories are Lévy flights.

We have also examined a second weakly turbulent flow, shown in Fig. 2(b). The arrangement of forcing holes was different, and there were no long-lasting coherent structures. (See Ref. [4] for details of the forcing for this flow.) Unlike the previous flow, particle trajectories wandered erratically and had no long flights. In Fig. 4, it can be seen that their angular displacements are much smaller than in the case with a coherent jet. No distinct flights can be seen, so we considered the distances particles moved in one direction before reversing direction. (As in the case with four vortices, we consider motion in the angular direction only.) The PDF for these "flight" motions, shown in Fig. 5(b), decays exponentially rather than with a power law. All moments of an exponentially decaying function are finite, so these are not Lévy flights.

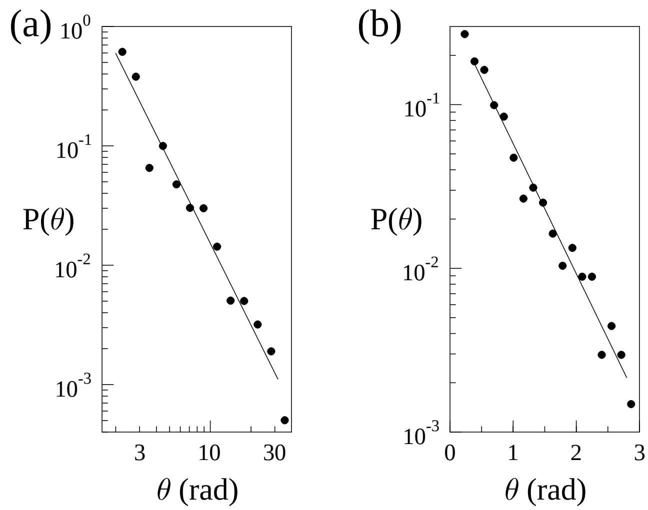

FIG. 5. Flight time probability distribution functions for (a) the ordered four vortex flow shown in Fig. 2(a) and (b) the weakly turbulent flow shown in Fig. 2(b). In (a), the log-log plot of the PDF has a power law decay $P(l) \sim l^{-\mu}$ with $\mu = 2.3 \pm 0.1$. In (b), the semi-log plot of the PDF shows exponential decay with a decay length of 0.35 ± 0.05 rad.

III. THEORY

For Lévy flights, the variance is expected to grow superdiffusively, $\sigma^2 \sim t^\gamma$, with $1 < \gamma < 2$. Calculations connect the flight PDF decay exponent μ with the variance exponent γ. Assuming random walkers move at constant velocity and take flights with a length PDF that decays as $P(l) \sim l^{-\mu}$, it has been shown [6,21,27,28] that

$$\begin{aligned} \gamma &= 2, & 1 < \mu < 2, \\ \gamma &= 4 - \mu, & 2 < \mu < 3, \\ \gamma &= 1, & \mu > 3. \end{aligned} \quad (4)$$

Exponential PDFs [such as Fig. 5(b)] have $\gamma = 1$, by the Central Limit Theorem. The case of $\gamma = 2$ is termed ballistic motion (rather than superdiffusive), and can be understood by considering the flight PDF. For $1 < \mu < 2$, the *first* moment of this PDF is infinite as well as the second moment. The mean flight length is infinite, so at any time the average random walker is still in the midst of its first flight[1]. As the random walkers are moving in different directions, the collection of random walkers moves apart from the origin at constant velocity and thus the growth is ballistic, like particles from a bomb, rather than diffusive, like spreading dye. The case for $2 < \mu < 3$ is intermediate between the ballistic and normal cases, and thus it is reasonable that the motion is superdiffusive and depends on μ. (Note that while our experiment considers one-dimensional random walks in the angular direction, these results are general and apply to random walks of higher dimensions [6].)

We have measured the variance for all of the observed particles for our two flows, as shown in Fig. 6. The ordered four vortex flow shows superdiffusive behavior, while the weakly turbulent flow shows normally diffusing behavior. A more detailed calculation, taking into account the time spent in vortices as well as the fact that all flights occur in the same direction, predicts $\gamma = 1.4$ for the ordered flow [6]. For both flows shown, a precise determination of γ from the data is impossible because of the lack of data for long times. Assuming that the weakly turbulent flow is normally diffusive at long times, a diffusion constant can be estimated as $D = 9$ cm^2/s [29]. Diffusion due purely to Brownian motion of our tracer particles would have a diffusion constant $D = 4 \times 10^{-12}$ cm^2/s.

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[1]To picture a PDF with an infinite first moment, consider the following case. A random walker takes a step of length 2 with probability 1/2, length 4 with probability 1/4, and generally length 2^n with probability $1/2^n$. Thus any particular step has a finite length, but the *average* step has length $\langle l \rangle = (1/2)(2) + (1/4)(4) + (1/8)8 + ... = \infty$. This corresponds to the case of a PDF with $\mu = 2$.

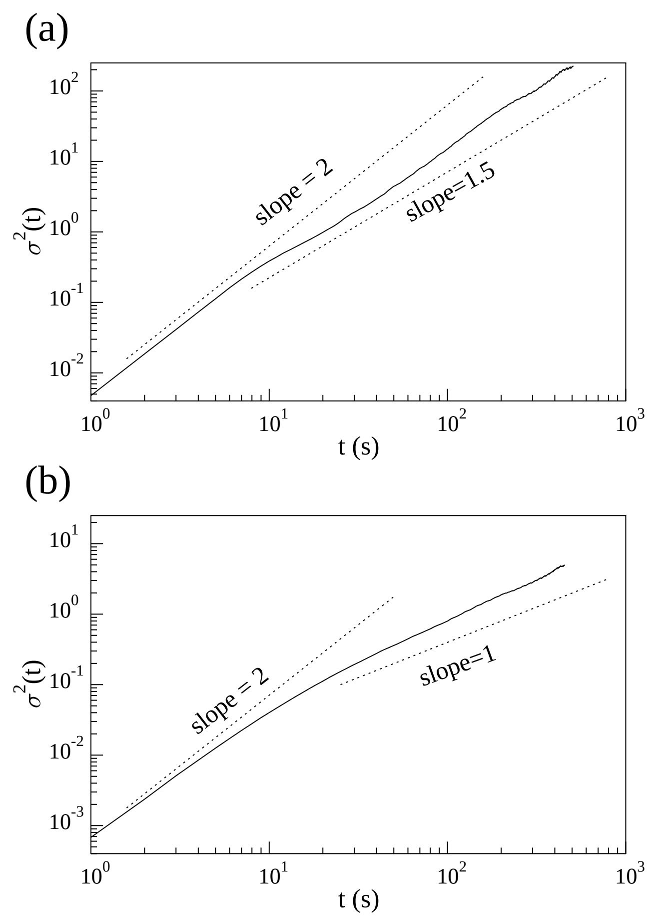

FIG. 6. Variance $\sigma^2(t)$ for (a) the four vortex flow and (b) the weakly turbulent flow. In both cases, the early behavior ($t < 10$ s) is ballistic ($\sigma(t) \sim t^2$), as all particles appear to be moving at a constant velocity and their motions are correlated [30]. For longer times, the structure of the flow influences the particle behaviors, and a crossover is seen to superdiffusive behavior in (a) and what may be normally diffusive behavior in (b).

IV. CONCLUSION

Our experiments are the first direct observations of tracer particles undergoing Lévy flights. Such observations allow us to directly measure the flight probability distribution functions. For the ordered four vortex flow shown in Fig. 2(a), we find the flight PDF decays as $P(l) \sim l^{-\mu}$ with $\mu = 2.3 \pm 0.1$, showing that the tracer particles are moving in Lévy flights. We have found this power law decay for flight PDFs to be a robust feature of flows with jets and vortices, with μ in the range 1.9–3.3 [3–6]. (For $\mu > 3$, the trajectories are no longer Lévy flights.) For all flows without jets, such as the weakly turbulent flow pictured in Fig. 2(b), the flight PDFs decay exponentially.

We hope that our work will motivate a search for the origins of the power law decay seen in our experiments. Perhaps future researchers will find similar behavior in jets in the atmosphere and oceans.

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