Mechanical Resonance and Chaos

You will use the apparatus in Figure 1 to investigate regimes of increasing complexity.

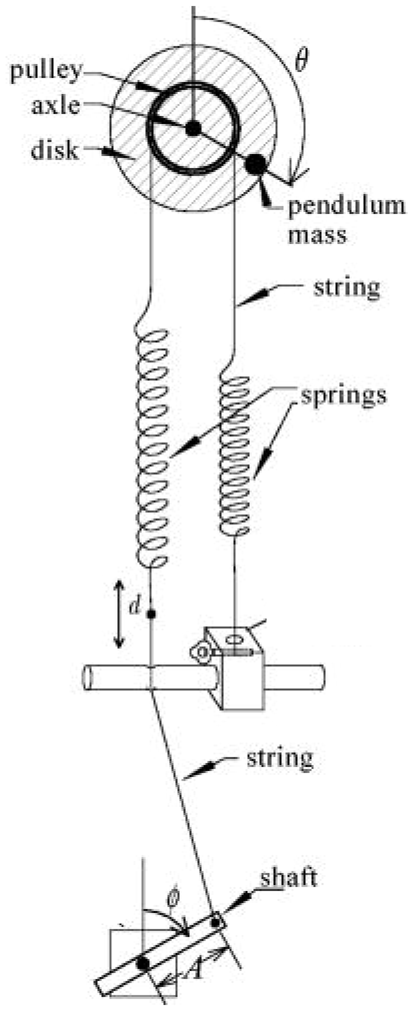

Figure 1. The rotary pendulum (from DeSerio, www.phys.ufl.edu/courses/phy4803L/group_IV/chaos/Chaos.pdf).

1. The Harmonic Oscillator

Retract the damping magnet as much as possible, and remove the pendulum mass. Now, the rotating disk is radially symmetric, and gravity exerts no torque it. The only torques are due to the forces of the strings (determined by the tensions in the springs).

Define θ as the clockwise displacement of the disk from its equilibrium position. Define x_e as the elongation of each spring when the disk is at equilibrium. At equilibrium, then, each spring stores potential energy $\frac{1}{2}kx_e^2$, where k is the spring constant. When the disk rotates clockwise from equilibrium through an angle θ, the elongation of the spring on the right decreases to $x_e - r\theta$, where r is the radius of the pulley; the elongation of the spring on the left must increase to $x_e + r\theta$. Thus, the potential energy stored in the spring on the right is $\frac{1}{2}k(x_e - r\theta)^2$, and the potential energy stored in the spring on the left is $\frac{1}{2}k(x_e + r\theta)^2$. The total potential energy is therefore

$$V = \frac{1}{2}k(x_e - r\theta)^2 + \frac{1}{2}k(x_e + r\theta)^2 = kx_e^2 + kr^2\theta^2. \quad (1)$$

Our apparatus measures θ and ω (dθ/dt); we cannot measure V directly. However, total mechanical energy is conserved: V + K = C, where K is the kinetic energy and C is a constant. Thus, V = C – K, where $K = \frac{1}{2}I\omega^2$, and I is the moment of inertia. Then

$$V = C - \frac{1}{2}I\omega^2. \quad (2)$$

Since we measure θ and ω, we can plot $-\omega^2$ vs. θ to determine the shape of V vs. θ. We can then compare our experimental plot with Equation (1).

We can derive the torque $\tau = -dV/d\theta$ from Equation (1):

$$\tau = -2kr^2\theta. \quad (3)$$

Theoretical tasks:

1.A. Use Equation (3) to find expressions for θ(t) and ω(t). What is f_0, the frequency of motion? Combine θ(t) and ω(t) to eliminate time: this result tells you what a phase plot (ω vs. θ) should look like.

At every moment in time, θ and ω each have a single value, and the system is completely characterized by the "phase point" (θ, ω). Since θ and ω change continuously with time, the phase point is always moving through phase space. The path of (θ(t), ω(t)) through phase space is called the trajectory.

1.B. In what direction (clockwise or counterclockwise) does the phase point move? Why?

Experimental tasks:

1.C. Use the Potential Well file to plot V vs. θ. First, screw the magnet all the way back to eliminate damping. Then, displace the disk from its equilibrium position and release. Click Start to record a few oscillations, and then click Stop. How does your plot compare with Equation (1)?

1.D. Plot θ vs. t and ω vs. t on the same axis. How do these curves compare with the theoretical equations of motion you derived from Equation (2)? What is the phase shift between θ and ω? Is this what you'd expect? Is the ratio between their amplitudes what you'd expect? What is the frequency f_0? Knowing f_0, r, and I (determined from measuring the mass and radius of the disk), estimate k. Measure k directly by suspending a known mass from a spring; compare your two values for k.

1.E. Generate a phase plot (ω vs. θ). What is the shape? What do you expect from the theoretical result derived from Equation (3)?

2. The Damped Linear Oscillator

When the magnet is brought near the rotating disk, eddy currents produce a damping torque. We assume that the damping torque is proportional to the angular velocity: $\tau = -b\omega$, where b is some constant. Then the total torque is

$$\tau = -2kr^2\theta - b\omega, \qquad (4)$$

which replaces Equation (3) as the equation of motion.

Theoretical task:

2.A. Find the solution to Equation (4). You may assume that it takes the form $\theta = Ce^{-at}\cos(2\pi ft)$, where C, a, and f are constants. By substituting this form into Equation (4), you can determine a and f in terms of the physical parameters of the system. Is f different from f_0, the frequency of the undamped oscillator?

Experimental task:

2.B. Position the magnet about half a centimeter from the rotating disk. Using the Chaos file, plot θ vs. t. How does this curve compare with the theoretical function you derive from Equation (4)? Determine b from the decay constant of the exponential envelope of θ vs. t. Repeat this for several values of b (several positions of the magnet). Perform 3.B below for each position of the magnet. (In other words, do both 2.B and 3.B before repositioning the magnet.)

3. The Damped, Driven Linear Oscillator

When the motor is turned on and the driver shaft rotates with frequency F, the position of the point d varies as $A\cos\phi = A\cos(2\pi Ft)$. Referring to Figure 1, the spring on the left is stretched more when this displacement is negative, and it is stretched less when the displacement is positive. Thus the elongation of the spring on the left becomes $x_e + r\theta - A\cos(2\pi Ft)$, and the torque due to the spring on the left acquires a new term: $rkA\cos(2\pi Ft)$. Adding this term to Equation (4) yields

$$\tau = -2kr^2\theta - b\omega + rkA\cos(2\pi Ft). \qquad (5)$$

The solution to Equation (5) consists of two terms. One of these terms, the "homogeneous solution," is the solution to Equation (4). As we've seen, this homogeneous solution decays to zero over time. The other term, the "inhomogeneous solution," retains its amplitude indefinitely. You may assume that the inhomogeneous solution takes the form

$$\theta = C\sin(2\pi Ft) + D\cos(2\pi Ft). \qquad (6)$$

By substituting Equation (6) into Equation (5), you can determine C and D in terms of the physical parameters. Next, use the identity $u\sin x + v\cos x = (u^2 + v^2)^{1/2}\cos[x - \tan^{-1}(u/v)]$ to rewrite Equation (6) in the form

$$\theta = \theta_{max}\cos(2\pi Ft - \psi). \qquad (7)$$

The steady-state amplitude θ_{max} does not depend on initial conditions. It only depends on the driving frequency and other physical parameters. You can manually immobilize or rotate the disk, but as soon as you it let go, it will gradually settle into the same steady-state motion (amplitude and phase shift) that it had before.

Theoretical task:

3.A. Determine θ_{max} as a function of physical parameters and F. Generate a theoretical plot of θ_{max} as a function of driving frequency F. You can assign values to the other parameters.

Experimental tasks:

3.B. Turn on the motor. Wait until steady-state motion is established: after some initial settling in, the path through phase space will retrace a single closed loop indefinitely. This loop is called the attractor for this system. Record the frequency F and the amplitude θ_{max}. Repeat for at least a dozen frequencies, concentrated near frequencies at which you predict the greatest amplitudes. Experimentally, which driving frequency yields the greatest amplitude? (This is called the resonance frequency.) Do your results agree with your theoretical θ_{max} vs. F plot? Repeat 3.B for several values of b.

The phase plot we examine is ω vs. θ. Actually, a third coordinate is needed to completely characterize the system: ϕ, the changing angle of the driver shaft. The phase plot we've seen shows ω vs. θ without providing any information about ϕ. Sometimes we like to look at "snapshots" of the ω vs. θ trajectory for a single value of ϕ. Our apparatus achieves this by recording whenever the driver arm crosses the photogate: this occurs only at a single value of ϕ (call it ϕ_p). The values of θ and ω at this moment are recorded and plotted as a point (θ, ω). The next time the driver arm crosses the photogate ($\phi = \phi_p$), θ and ω are again recorded and plotted. The plot of these points is called the Poincare section.

3.C. Superimpose the Poincare section on the phase plot during steady-state motion. Does this make sense? What would the Poincare section look like if the motion were not periodic?

4. The Nonlinear Oscillator

Now let's add the "point mass" to the rotating disk and temporarily remove damping and driving. The point mass changes the moment of inertia of the rotating system and also contributes a torque $mgL\sin\theta$, where L is the distance from the point mass to the axis. The equation of motion is now

$$\tau = -2kr^2\theta + mgL\sin\theta. \qquad (8)$$

Theoretical tasks:

4.A. Determine the potential energy $V(\theta)$. Under what conditions are there two minima? Write an expression for these two minima, $\pm\theta_0$. (There is no closed-form solution.)

4.B. Do a Taylor series expansion around $\theta = \theta_0$. Keep terms through second order. In other words, use

$$V(\theta) \approx V(\theta_0) + (\theta - \theta_0)\left.\frac{dV}{d\theta}\right|_{\theta_0} + \frac{1}{2}(\theta - \theta_0)^2 \left.\frac{d^2V}{d\theta^2}\right|_{\theta_0} .$$

Next, use $\tau = -dV/d\theta$ to derive an approximate equation of motion.

4.C. Using your result for 4.B, determine θ as a function of t for small oscillations around θ_0. What is the frequency of the motion?

Experimental tasks:

4.D. Turn off the motor and screw the magnet all the way back. Identify the hole where you will attach the point mass. By rotating the driver arm, position this hole as closely as possible to $\theta = 0$ when the disk is at equilibrium. Screw in the point mass. Observe that there are now two positions of stable equilibrium (potential wells), one on the left, and one on the right. Displace the disk enough so that the point mass travels through both potential wells. Open the Potential Well file. Click Start to record a few oscillations, and then click Stop. How does your plot compare with your result for 4.A? How far apart are the two minima experimentally and theoretically?

4.E. It's hard to solve Equation (8) for the nonlinear oscillator! So let's study small oscillations near the minima; for θ near θ_0, the Taylor series expansion is approximately accurate. Record θ vs. t for small oscillations in one of the potential wells. What is the frequency? Is it constant? How does it compare with your prediction in 4.C?

5. The Damped, Driven Nonlinear Oscillator

In general, there are no closed-form solutions for the damped, driven nonlinear oscillator. However, we can observe a variety of periodic and chaotic oscillations. If a single loop appears in the phase plot, the motion is called period-1. This is what you observed in 3.B. If the trajectory appears to cross over itself to form a smaller loop within a larger loop, the motion is called period-2. If N loops appear in the phase plot, the motion is called period-N. If the trajectory never repeats itself, the motion is chaotic. In some systems, adjustment of one parameter (for example, driving frequency) creates period doubling: the motion changes from period-1 to period-2 to period-4 to period-8, etc. Eventually, the motion becomes chaotic.

Experimental tasks:

5.A. Try to observe three types of period-1 oscillations: oscillations within the well on the left, oscillations within the well on the right, and oscillations from one well to the other. Adjust damping, driving frequency, and initial conditions as needed.

5.B. After you've found stable period-1 oscillations, slowly adjust the driving frequency and watch for any period-2 oscillations. Record the frequency at which you observe a transition from period-1 to period-2. If you see period-3, period-4, etc., record the transition frequencies. If you see a transition to chaos, record this also.

5.C. Use the Bifurcation file to record the angular positions at which the angular velocity is zero; these are the positions at which the disk's rotation switches directions. There are two such positions for period-1, four such positions for period-2, and 2N such positions for period-N. Slightly increase the frequency every thirty seconds or so. Discuss the transitions indicated in your plot of "turn around" positions vs. time. Repeat for a few damping intensities.