Elasticity of Metallic Spheres

I. GOAL

The goal in this lab is to test the Hertzian theory of elasticity [1], and determine the elastic modulus E, the elasticity, of stainless steel. You accomplish this by measuring how much large steel spheres deform upon impact collisions of varying strength. The first part of the lab involves building a digital timing circuit needed to measure the collision times, which is then used in the experimental measurements in the second part. Your results will then be compared to theory, where a determination of E can be made.

II. INTRODUCTION AND THEORY

We need to derive an expression for the collisional contact time τ for an impacting sphere in terms of the sphere's speed U_0, mass M, radius R, density ρ and elasticity E. We begin by writing down the solution for the force F(x) needed to compress the sphere a distance x. The solution comes from the Hertz model [1, 2] for small deformations of elastic objects, and is a rather long calculation owing to the complex geometry of spherical deformations. The result is

$$F(x) = \frac{2E\sqrt{R}}{3\left(1-\sigma^2\right)} x^{3/2} \tag{1}$$

Here σ is called the Poisson ratio [3], which for most metals has a value of $\sigma \approx 0.3$, and we assume $\sigma = 0.30$ in what follows below. Equation (1) is reminiscent of Hooke's linear law for springs, $F = -k \cdot x$, except that the power of the compression variable x is higher than 1, i.e. $F \propto -x^{3/2}$. This means that the system is non-linear, i.e. if we double the compression, we more than double the force. Can you explain why this would be the case for spheres, but not springs? Think about the size of the region that gets compressed as a function of x. From the force expression Eq. (1) we now derive a number of useful relations that are relevant for the experiments.

1. <u>Maximal compression distance h:</u>

   How much will a sphere compress (Fig. 1) when colliding with a plate? To determine this, you can use an energy conservation argument: A sphere whose decrease in height is H will have a velocity $U_0 = (2gH)^{1/2}$ when it first contacts the plate. Its kinetic energy then is $KE_0 = (1/2)MU^2$. There is also a potential energy of compression in the steel ball. Using Eq. (1), derive a general expression for PE(x). Now, assuming that all of the initial kinetic energy goes into compressing the sphere, show that the maximal compression distance h can be written as

   $$h = 2.20\rho^{2/5}RU_0^{4/5}/E^{2/5}. \tag{2}$$

   In the experiments we will not be able to directly measure h, because it is extremely small for steel. However, we will measure the collision time τ, and the expression for h here will be useful in its derivation below.

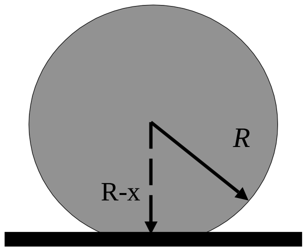

FIG. 1: Diagram of a compressed elastic sphere. The compression distance is x, so that the centerline radius has been reduced from R to $R - x$.

2. <u>Compression time τ:</u>

To derive an expression for the total compression time τ requires several parts, and begins from Newton's basic force balance law. That is, write out $M(dU/dt) = -F(x)$ using the force expression in Eq. (1).

- Let's first solve for the sphere velocity $U(x)$ in terms of the compression distance x. To do this, we need to eliminate time as a variable, which we can do with the identity $\frac{dU}{dt} = \frac{dU}{dx}\frac{dx}{dt} =$

  $U(dU/dx)$. Now separate the U and x terms onto different sides of the equation, and integrate the expression from U_0 to $U(x)$ on $U-$side, and 0 to x on the $x-$side. Show that the velocity $U(x)$ as a function of compression distance can be written as

$$U(x) = U_0\sqrt{1 - \left(\frac{x}{h}\right)^{5/2}}. \tag{3}$$

- To solve for the compressional time, write $U = \frac{dx}{dt}$ in Eq. (3), and separate the x and t terms onto different sides of the equation. The total time that the sphere is in contact with the plate, τ, is assumed to be twice the time that it takes the sphere to go from no compression ($x = 0$) to full compression ($x = h$). To solve for τ, integrate from 0 to h on the $x-$side, and 0 to $\tau/2$ on the $t-$side (refs. [5], [6] can help with the integration), and show that $\tau = 2.94h/U_0$, or substituting in for h from Eq. (2),

$$\tau = 6.46\rho^{2/5}\frac{R}{U_0^{1/5}E^{2/5}}. \tag{4}$$

  This is our first result that can be directly used in the experiments. As will be outlined below, by measuring τ for different values of R and U_0 (i.e. different dropping heights), we can extract values of the elasticity of steel E.

3. <u>Contact area A:</u>

The contact area A made between the sphere and the plate is also a very useful quantity to know, and it can be readily measured. To do this, you will lay a thin piece of aluminum foil on top of the base plate, and examine the area A of the indented foil after the collision. What does A tell us? The measured contact area corresponds to the maximum contact area between the sphere and the plate. Most importantly, by geometric arguments [4], one can show that A is directly related to the maximum compression distance h, as

$$A = \pi R h. \tag{5}$$

Therefore, while we cannot measure the compression distance h directly, by measuring A, we are able to estimate its value from the measurements of A.

4. <u>Expressions relating contact area and collision time</u>

Finally, we can construct from the above Eqs. (2,4,5) two interesting and useful relations that relate contact area and collision time. These are

$$A/(\tau U_0) = 1.069R \tag{6}$$

$$A^{1/2}\tau^2 = 110\rho R^3/E \tag{7}$$

III. LAB INSTRUCTIONS

The lab consists of two general parts. The first part is the construction of a digital timing circuit to be used in the second part for measurements of contact times.

A. Construction of a Digital Timing Circuit.

The contact times that we would like to measure are on the order of 100 μs. Obviously, this is no task for a stopwatch. We need an instrument with a precision of about 1 μs. Perhaps such an instrument is commercially available; if it is, it's likely to be expensive. A straightforward and instructive alternative to purchasing an instrument is to build our own.

The circuit we are going to construct has three main components: a clock oscillator, counters, and hexadecimal displays. These produce the following effects. The clock oscillator generates one pulse every microsecond. During our experiment, the contact between the ball and the plate allows these microsecond pulses to travel to the first counter; the ball and the plate act as a switch. When a pulse arrives at the counter, its output increases by 1.

Each counter has four output pins whose voltage can take on only two values (see ref. [7]). The high value is represented by 1, and the low value is represented by 0. The four output pins are associated with the digits ("bits") of a binary number, so a total of 16 binary numbers can be represented by four pins: 0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, and 1111. We want to be able to count higher than 16, so we connect the highest bit (the most slowly changing bit) of the first counter to the input of the second counter. This lets us count up to 16^2, but that's not high enough. To count up to $16^3 = 4096$, we connect the highest bit of the second counter to the input of the third counter. So our circuit will be able to measure time intervals as high as 4096 μs; each microsecond pulse increases our binary number by 1.

Finally, we need an easy way to read off the values stored in our counters. We simply connect each counter to a hexadecimal display (see ref. [8]). The display shows us the hexadecimal digit equivalent to the number stored in the counter. In hexadecimal, every whole number through 15 is represented by only one digit: 10 is represented by A, 11 by B, 12 by C, 13 by D, 14 by E, and 15 by F. We have one hexadecimal display for each counter. It's convenient to order the three displays from left to right to obtain a three-digit hexadecimal number. The display on the right must connect to the counter storing the lowest bits (which I've called the first counter), and the display on the left must connect to the counter storing the highest bits. If we see D8A, we compute $13 \times 162 + 8 \times 16 + 10 \times 1 = 3466$, and we know that we've measured a time interval of 3466 μs.

B. Measurements

The equipment needed for this experiment is:

- 4 steel balls.
- mass scale to weigh steel balls.
- small ruler.
- yardstick ruler.
- sheets of aluminum foil.
- completed electronic timing circuit.

Experimental Procedure:

Each ball should be released from different heights. The contact time τ from the digital circuit should be recorded, as well as the impact area A imprinted in the aluminum foil sheet. Multiple runs should be done for each ball, at each height, to check reproducibility and to estimate the experimental uncertainty.

IV. REPORT

The report should contain at a minimum the following items:

- A description of the construction and operation of the digital timing circuit.
- A description of the elastic Hertz model introduced in Sec. II, including explicit derivations of Eqs. (2-4) as described in the text.
- Explicit tests of the Hertzian elasticity model using your data, specifically testing the functional form of the contact time and contact area.
- Can you get out reasonable values for the Young's modulus of steel from fits to the τ and A plots?
- An evaluation of Eqs. (6) and (7). Does Eq. (6) yield accurate values of the sphere radii?
- Calculate the range of compression distances h, from A, seen in the experiments.
- What are the important sources of uncertainty in this lab?

V. REFERENCES

[1] L.D. Landau and E.M. Lifshitz, Course of Theoretical Physics, (Pergamon, Oxford), Vol. 7 , 26-31, (1959).
[2] B. Leroy, Collision between two balls accompanied by deformation: A qualitative approach to Hertz's theory, Am. J. Phys. 53, 346-349 (1985).
[3] http://en.wikipedia.org/wiki/Poisson_ratio
[4] http://www.oxfordcroquet.com/tech/gugan/index.asp#refcite=R1
[5] http://integrals.wolfram.com/index.jsp
[6] http://functions.wolfram.com/webMathematica/FunctionEvaluation.jsp?name=Hypergeometric2F1
[7] http://focus.ti.com/lit/ds/symlink/sn74ls93.pdf.
[8] http://focus.ti.com/lit/ds/symlink/til311.pdf